Heterogeneous Beliefs and the Cross-section of Stock Returns

Bingcun Dai*   Mungo Wilson†

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Abstract

We develop a mean-variance model of capital market equilibrium with heterogeneous beliefs. When investors disagree about future asset prices, optimists and pessimists wish to borrow cash and assets, respectively, from each other, to take positions in line with their beliefs. Borrowing contracts are collateralized by borrowers' wealth. However, due to disagreement, lenders do not value the collateral as much as their borrowing counterparts do, thus are more reluctant to lend than borrowers are eager to borrow. This limits the borrowing capacities of investors, which endogenously generates leverage and short-sales constraints. Under a model mapping consensus belief to this heterogeneous equilibrium, we derive a modified CAPM that incorporates the shadow prices of the two borrowing constraints, which appear with opposite signs. With either very low or very high disagreement, the standard CAPM holds because both constraints either do not bind or bind to a similarly large degree. When disagreement is moderate, the leverage constraint is generally tighter than short-sales constraint, although both bind. Consequently, less risky assets will be underpriced relative to the standard CAPM, while riskier assets will be overpriced. We

*bingcun.dai@sbs.ox.ac.uk. Said Business School, Oxford University.
†Mungo.Wilson@sbs.ox.ac.uk. Said Business School, Oxford University, and Oxford-Man Institute.

confirm our theoretical predictions using the daily trading volume in the stock market as our measure of disagreement.

I. Introduction

The capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965) implies a cross-sectional relation between expected return and risk: expected returns on securities are a linear function of their market betas, and all cross-sectional variation in returns can be fully captured by betas, i.e. only systematic risk is priced. Thus, the relationship between risk and expected return is plotted on a single security market line (SML). However, there is strong evidence that the empirical SML differs significantly from that predicted by the CAPM: The intercept is too high and the slope is too low, even negative for very high-beta assets. The slope of the SML can be affected by certain factors such as the level of inflation (Cohen, Polk, and Vuolteenaho (2005)), investors' demand for lottery-like stocks (Bali, Brown, Murray, and Tang (2017)), investor sentiment (Antoniou, Doukas, and Subrahmanyam (2016)), types of information events (Savor and Wilson (2014)), leverage constraints (JylhÄ (2018)), and short-sales constraints (Hong and Sraer (2016)).

Some theoretical studies try to reconcile the observed deviations from the predicted SML by relaxing the assumptions underlying the standard CAPM. One essential assumption is that investors can take long or short positions of any size in any asset, including the risk-free asset. Black (1972) shows that when investors are unable to borrow at the risk-free rate, the slope of the SML is smaller than in the unconstrained CAPM. Frazzini and Pedersen (2014) extend the Black (1972) model under a less restrictive condition that investors are allowed to leverage but face margin requirements, and show that the slope of the SML decreases in the tightness of the leverage constraint. In models with restricted borrowings, constrained investors who desire high returns cannot lever up the tangency portfolio, and thus have to increase their holdings in riskier assets (high-beta stocks). The increase in demand for high-beta stocks bids up prices, and in turn flattens the SML. Although those models yield a flatter SML relative to the CAPM, they cannot explain the evidence that high-beta stocks deliver lower returns than low-beta stocks, which implies a downward-sloping SML (Black (1972), Baker, Bradley, and Wurgler (2011), Frazzini and Pedersen (2014)).

On the other hand, to obtain a downward-sloping SML, Hong and Sraer (2016) construct a model with short-sales restrictions, i.e. constraints on borrowing risky assets rather than the risk-free asset. Moreover, they relax the other CAPM assumption of complete agreement and allow agents to disagree about the common component of cash flows. High-beta assets are more sensitive to these aggregate disagreements and thus experience greater divergence of opinion. Thus, pessimists, who want to but are unable to take short positions in high-beta assets, are sidelined due to these short-sales constraints. As a consequence, such assets are held only by optimists and hence overpriced compared to low-beta assets.

We find supporting evidence for the implication of the Hong and Sraer (2016) model, i.e. expected returns decrease with beta for high-beta assets (Figure 3). However, our finding is at odds with their implication when disagreement is very high: on days with very high disagreement (defined as the top decile of the market turnover), the SML runs very near to its CAPM prediction (Figure 4), contrary to the Hong and Sraer (2016) 's prediction of a downward-sloping SML. In other words, the market portfolio is *efficient* when disagreement is very high. If short-sales constraints lead to the inefficiency of the market portfolio (i.e. deviation from the standard CAPM), it is a *puzzle* that, when the stock market experiences extremely high disagreement and therefore short-sales constraints are more binding, the market portfolio is instead efficient.

One drawback of all of the models above is that they take leverage and short-sales constraints as exogenous, as well as imposing either one or the other, but not both, when in fact these constraints arise simultaneously in equilibrium from, inter alia, disagreement between agents. Take a simple example: investor A values a risky asset at $3 while investor B values the same asset at $12. If the market price is $7.50 (the consensus belief), B wishes to borrow and invest more in the asset, pledging the asset as collateral. However A values this collateral at $3 and is therefore reluctant to lend B the full amount she wishes to borrow, worrying that B may default if A's beliefs turn out to be correct. If A is the only lender out there, then B will be leverage-constrained by A's different beliefs. A similar argument shows that A can be short-sales constrained by B's different beliefs, although not in this example.

In addressing the *"efficient frictions" puzzle* described above, this paper offers a theoretical framework incorporating heterogeneity and borrowing constraints on both risky assets and risk-free assets, i.e. short-sales constraints and leverage constraints. We model investors as being heterogeneous in their beliefs about expected payoffs and the covariance matrix of risky assets, as in Lintner (1969) and Jarrow (1980), and derive their willingness to lend cash or shares to investors with different beliefs. Because default is analytically rather complex to work with, we instead use a model in which investors are subject to margin requirements designed to prevent default. In equilibrium there is no default, but disagreement between investors makes margin contraints bind more tightly. Investors can use leverage (or short selling), but this margin requirement dictates the minimum value of collateral (i.e. part of investors' wealth) needed in a borrowing contract, limiting the maximum amount of borrowing. In addition to being realistic, these margin constraints are analytically tractable.

Our main result is that for an investor with average (consensus) beliefs, the expected excess return on a risky asset relates to its expected market beta in the linear form as the standard CAPM, plus an extra term, which is related to the shadow price difference between leverage constraints and short-sales constraints. This extra term, defined as the *distance-to-CAPM,* explains the variation of the SML for different levels of disagreement. In particular, when leverage constraints and short-sales constraints bind to a similar degree, the *distance-to-CAPM* is close to zero, so that the SML runs very near to the standard CAPM prediction, which rationalizes the *"efficient frictions" puzzle.* The intuition is as follows. The leverage-constrained investors who have a preference for high return will seek to leverage riskier (high-beta) stocks instead of the tangency portfolio. Similarly, the short-sales constrained investors will take more short positions in high-beta stocks to achieve a higher expected return. When the price pressure on high-beta stocks from leveraging is balanced by that from short selling, the stocks are efficiently priced as if there were no borrowing frictions.

The reason why the "efficient frictions" can be observed only when disagreement is very high is that the disagreement-induced leverage constraint is generally tighter than the corresponding short-sales constraint for a given level of disagreement. Returning to our simple example, assume agent A has \$7.50 of cash and agent B has 1 share with a market price of \$7.50 under consensus beliefs. Agent B, the optimist, wants to borrow \$7.50 from A, expecting to be able to repay \$7.50 (ignore discount rates), buy one additional share and make a profit of \$4.50 (\$12 - \$7.50) from her cash borrowing. If she succeeds she can own two shares and post them as collateral. But agent A believes the shares will only be worth \$3 each and is therefore only willing to lend \$6, making B leverage constrained to the amount of \$1.50 given agent B's beliefs. Agent A wants to borrow one share from B and sell it today for \$7.50, expecting to be able to deliver back 1 share tomorrow and make a \$4.50 profit. Note agent A's sale of her own share plus the sale of the share borrowed from B yields a \$15 cash payment today she can post as collateral. Agent B expects the price to go to \$12 so that A will have enough collateral (\$15) to repurchase the share she is short, and is therefore willing to lend her share to A. Thus A is not short-sale constrained. Thus despite equal disagreement relative to consensus and equal initial wealth, the optimist who wants to leverage is constrained given her beliefs, but the pessimist who wants to short-sell is not. The reason is that, in this example, the cash-borrower's collateral is riskier than the share-borrower's collateral.

A second reason why the leverage constraint binds more tightly in general is that agents tend to default on cash borrowings when share prices are low (bad times), whereas short-sellers tend to default on their share borrowings when share prices are high (good times). Since all agents are risk-averse, agents dislike default in bad times more than they dislike default in good times.

The source of the endogenous borrowing constraints is that lenders want to avoid the potential loss that they would suffer if they satisfied borrowers' demand. Pessimistic lenders of cash worry, that if borrowers default and give up the collateral (i.e. stocks), the drop of stock price will lead to a real reduction in their wealth, all of which can be lost when the stock is worthless. In contrast, the nominal value of the collateral for optimistic lenders of stocks is guaranteed. Thus, the potential loss that optimistic lenders care about is not a real loss with a negative net income; instead, it is like a loss of a lottery prize for betting correctly. Therefore, the potential loss for pessimistic lenders is of more real impact, and thus they will impose a tighter constraint on borrowers than optimists do. The investors who hold biased beliefs may bid up high-beta stock prices to the point of having lower returns than low-beta stocks, which implies a downward-sloping SML. When belief dispersion is very high, there is almost no lending by pessimists to optimists, which leads to maximum tightness of borrowing constraints, i.e. leverage constraints and short-sales constraints will bind to a similarly large degree.

Our model also implies that heterogeneous beliefs command a positive equity premium, consistent with the findings in Anderson, Ghysels, and Juergens (2005), David (2008) and Banerjee and Kremer (2010). We find that the shadow price difference between leverage constraints and short-sales constraints, which results endogenously from heterogeneous beliefs, is a determinant of the equity premium. Two constraints have opposite effects on asset prices. On the one hand, leverage-constrained investors would like to borrow cash and invest in equity but the constraint prevents them from doing so. Therefore, the leverage constraint decreases the demand for equity, and lowers the stock price (increases expected stock return) as a result. On the other hand, when the short-sales constraint is binding for some pessimists, equity prices will not reflect the valuations of those pessimists but reflect the valuations of relatively more optimistic investors, which pushes up stock price (lowers expected stock return). As discussed earlier, the leverage constraint is generally tighter than the short-sales constraint, and thus the net effect of disagreement is to increase expected stock returns overall.

Moreover, we show that the effect of disagreement on trading volume can be either positive or negative: trading volume increases in disagreement since investors with dispersed beliefs have high speculative demand; but disagreement also has a negative effect which limits the growth in trading volume, because the increase in disagreement tightens borrowing constraints and thus reduces the volume which would be traded relative to the volume if the constraints did not bind. Although it is difficult to obtain an analytical derivative of trading volume with respect to disagreement, given the evidence (Atiase (1991), Bessembinder, Chan, and Seguin (1996), Goetzmann and Massa (2005)), we could plausibly conjecture that the positive direct effect is stronger in general: trading volume increases in disagreement.

Our paper is strongly related to Simsek (2013), but whereas his paper consider only a single risky aset, our paper realtes to the entire cross-section. We derive implications for the entire SML. Neither of our papers microfounds disagreement, which would be an important extension of this paper. Our paper is also related to the work of Fostel and Geanakoplos, particularly Fostel and Geanakoplos (2015), particularly in its solution technique. Again, they do not attempt to derive results for the SML.

Our model yields the following key testable implications. (i) When disagreement is very low or very high, the standard CAPM holds because two constraints do not bind or bind to a similarly large degree. (ii) When disagreement is moderate, the effect of the leverage constraint dominates that of the short-sales constraint; the average leverage-constrained investor would strategically substitute high-beta stocks for low-beta stocks. Consequently, less risky assets will be underpriced relative to the standard CAPM (*distance-to-CAPM*>0), while riskier assets will be overpriced (*distance-to-CAPM*<0). When the net effect of the stock-borrowing constraint is large enough, the SML will be downward-sloping as in Hong and Sraer (2016).

Next, we conduct a preliminary empirical analysis to test our model predictions using the daily trading volume in the stock market as our measure of disagreement. Consistent with our theoretical predictions, we find that (i) on days of very low and very high stock market turnover, the empirical Security Market Line runs very near its CAPM prediction; (ii) in contrast, when turnover is not extreme, the empirical SML is above that predicted by the standard CAPM for low-beta assets and below it for high-beta assets.

The rest of the paper is organized as follows. In Section 2 we present the model. Section 3 describes the data and shows the empirical results. We conclude in Section 4.

II. Model

A. Securities Market

Consider a market with one risk-free rate asset and K risky assets. There are two periods, $t = 0$ and $t = 1$. At $t = 0$, let the price of the risk-free asset be 1 and the price of risky assets be $P = (p_1, \cdots, p_K)^T$. At $t = 1$, the payoff of the risk-free asset is $R_f = 1 + r_f$ and the payoff of risky assets are $X = (x_1, \cdots x_k, \cdots, x_K)^T$.

B. Investors

There are I investors in the market, indexed by $i = 1, 2, \cdots, I$. The investors differ in their initial endowments, risk aversion, and probability assessments of prices which are joint normally distributed. Let the belief of investor i represented by $B_i = \{E_i(X), \Omega_i\}$ be the expected payoff and the covariance matrix as given:

$$E_i(X) = (E_i(x_1), \cdots, E_i(x_K))^T$$

$$\Omega_i = (\sigma_{i,kl})_{K\times K}, \qquad \sigma_{i,kl} = cov_i(x_k, x_l).$$

At $t = 0$ investor i invests her initial wealth $W_{i,0}$ in a portfolio which contains $Z_i = (z_{i,1}, z_{i,2}, \ldots, z_{i,K})^T$ shares of risky assets and the risk-free asset. The wealth of the portfolio for investor i at $t = 1$ is

$$W_i = R_f\left(W_{i,0} - P^T Z_i\right) + X^T Z_i,$$

and thus

$$E_i(W_i) = R_f\left(W_{i,0} - P^T Z_i\right) + E_i^T(X) Z_i, \qquad \sigma_i^2(W_i) = Z_i^T \Omega_i Z_i. \tag{1}$$

Investor i has a risk aversion coefficient of γ_i and a constant absolute risk aversion (CARA) utility function $U_i(W_i) = -e^{-\gamma_i W_i}$. As W_i is normally distributed, the investor's optimal portfolio is obtained by maximizing the certainty-equivalent of future wealth:

$$V_i\left(W_i\right) = E_i\left(W_i\right) - \frac{\gamma_i}{2}\sigma_i^2\left(W_i\right)$$

Investors can borrow cash and stocks, but need to post an initial margin of m^L and m^S. Thus, the maximum investment in risky assets is $W_{i,0}/m^L$ and the maximum borrowing of risky assets is $W_{i,0}/m^S$.

C. Portfolio Choice

Investors choose portfolios by maximizing their date-1 utility under leverage and short-sales constraints. Thus the optimization problem is given by

$$\max V_i\left(W_i\right) = E_i\left(W_i\right) - \frac{\gamma_i}{2}\sigma_i^2\left(W_i\right)$$

subject to

$$m^L P^T Z_i \leq W_{i,0}$$

$$m^S P^T Z_i \geq -W_{i,0}$$

The Lagrangian is given by

$$\mathcal{L} \;=\; E_i\left(W_i\right) - \tfrac{\gamma_i}{2}\sigma_i^2\left(W_i\right) + \psi_i\left(W_{i,0} - m^L P^T Z_i\right) + \phi_i\left(m^S P^T Z_i + W_{i,0}\right)$$

where ψ_i and ϕ_i are non-negative shadow prices of the leverage constraint and the short-sales constraint, respectively.

Therefore the first order conditions are

$$\frac{\partial \mathcal{L}}{\partial Z_i} = \left[\frac{\partial E_i\left(W_i\right)}{\partial Z_i} - \frac{\gamma_i}{2}\frac{\partial \sigma_i^2\left(W_i\right)}{\partial Z_i}\right] - m^L\psi_i P + m^S\phi_i P = 0 \tag{2}$$

$$\psi_i \geq 0, \qquad m^L P^T Z_i \leq W_{i,0}, \qquad \psi_i\left(W_{i,0} - m^L P^T Z_i\right) = 0$$

$$\phi_i \geq 0, \qquad m^S P^T Z_i \geq -W_{i,0}, \qquad \phi_i \left(m^S P^T Z_i + W_{i,0} \right) = 0$$

Given the expression (1) we have

$$\frac{\partial E_i (W_i)}{\partial Z_i} = E_i (X) - R_f P \qquad \frac{\partial \sigma_i^2 (W_i)}{\partial Z_i} = 2 \Omega_i Z_i. \tag{3}$$

Substituting (3) into (2) leads to the optimal demand for risky assets

$$Z_i = \tau_i \Omega_i^{-1} \left[E_i (X) - \left(R_f + m^L \psi_i - m^S \phi_i \right) P \right] \tag{4}$$

where $\tau_i = 1/\gamma_i$ is the risk tolerance coefficient.

D. Equilibrium

Investors choose the optimal portfolio strategies subject to the equilibrium prices based on their own assessments of the joint distribution for risky asset prices. The equilibrium prices are achieved when the market clears and the sum of individual demands equals aggregate market supply:

$$\sum_{i=1}^{I} Z_i = Z_m \tag{5}$$

where Z_m is the number of shares outstanding of risky assets. To establish the result, we represent heterogeneous beliefs by a consensus belief which, if held by all investors, would result in the same equilibrium prices as in the actual heterogeneous economy, as given by the definition in Verrecchia (1979).

Definition 1 *The consensus belief about the expected payoff or covariance matrix of the risky assets is an average of individual beliefs, weighted by their risk tolerances and the inverses of the covariance matrices:*

$$\Omega^{-1} = \sum_{i=1}^{I} \frac{\tau_i}{\tau} \Omega_i^{-1} \tag{6}$$

$$E(X) = \Omega \sum_{i=1}^{I} \frac{\tau_i}{\tau} \Omega_i^{-1} E_i(X) \tag{7}$$

where τ is the aggregate risk tolerance $\tau = \sum_{i=1}^{I} \tau_i$.

Similarly define the aggregate shadow price of short-sales constraint and leverage constraint as

$$\Phi = m^S \Omega \sum_{i=1}^{I} \frac{\tau_i}{\tau} \Omega_i^{-1} \phi_i, \qquad \Psi = m^L \Omega \sum_{i=1}^{I} \frac{\tau_i}{\tau} \Omega_i^{-1} \psi_i. \tag{8}$$

We denote by r_k the return per share for asset k and by r_m the return on the market portfolio which is the value-weighted average of all risky assets. Then, under the consensus belief, the expected return of asset k is given by $E(r_k) = E(x_k)/p_k - 1$, and the market beta of asset k is $\beta_k = cov(r_m, r_k)/\sigma^2(r_m)$, where $cov(r_m, r_k)$ and $\sigma^2(r_m)$ are obtained from the covariance matrix Ω (6).

Lemma 2 *Under the consensus belief as defined in Definition 1, equilibrium prices are given by*

$$P = (R_f I_K + \Psi - \Phi)^{-1} \left[E(X) - \tau^{-1} \Omega Z_m \right] \tag{9}$$

and the optimal portfolio of agent i is

$$Z_i = \tau_i \Omega_i^{-1} \left[E_i(X) - \chi_i \left(E(X) - \tau^{-1} \Omega Z_m \right) \right]. \tag{10}$$

where $\chi_i = (R_f + \psi_i - \phi_i)(R_f I_K + \Psi - \Phi)^{-1}$, and I_K is an identity matrix.

PROOF: See Appendix A.

Proposition 3 [CAPM with heterogeneous beliefs] *Under the consensus belief in Definition 1, the expected excess return on a risky asset relates to its expected market beta in the linear form as the standard CAPM, plus an extra term, which represents the effect that borrowing constraints have net of short-sales constraints on the Security Market Line:*

$$E\left(r_k\right) - r_f = \underbrace{\beta_k\left[E\left(r_m\right) - r_f\right]}_{\textit{standard CAPM}} + \underbrace{f\left(\Psi - \Phi, \beta_k\right)}_{\textit{extra term}}, \tag{11}$$

where

$$f\left(\Psi - \Phi, \beta_k\right) = \frac{\left[\left(\Psi - \Phi\right)P\right]_k}{p_k} - \beta_k \frac{P^T\left(\Psi - \Phi\right)Z_m}{P^T Z_m}.$$

The extra term $f\left(\Psi - \Phi, \beta_k\right)$ can be naturally interpreted as the mispricing relative to the standard CAPM, defined as the distance-to-CAPM (DtC). If DtC is positive (negative), the empirical SML is above (below) that predicted by the standard CAPM, and assets are underpriced (overpriced) relative to the standard CAPM.

PROOF: See Appendix B.

Corollary 4 *In order to obtain cleaner empirical predictions, we assume that investors agree upon the covariance matrix. Then Ψ and Φ are reduced to scalars, and thus $f\left(\Psi - \Phi, \beta_k\right) = \left(\Psi - \Phi\right)\left(1 - \beta_k\right)$. Consequently, equation (11) becomes*

$$E\left(r_k\right) - r_f = \beta_k\left[E\left(r_m\right) - r_f\right] + \chi\left(1 - \beta_k\right), \tag{12}$$

where $\chi = \Psi - \Phi$ is the difference in shadow price between the leverage constraint and the short-sales constraint. The distance-to-CAPM is given by

$$DtC = \chi\left(1 - \beta_k\right). \tag{13}$$

Remark 5 *(i) When $\chi = 0$, i.e. leverage constraint and short-sales constraint bind to a similar degree, $DtC = 0$ and thus the standard CAPM holds; (ii) when $\chi \neq 0$, i.e. either constraint dominates, DtC will flip the sign around $\beta = 1$. For example, when χ is positive, DtC will be positive for less risky assets ($\beta < 1$), and negative for riskier assets ($\beta > 1$). Therefore, χ, the net effect of borrowing constraint, is like a force that rotates the SML around the the fixed point which represents the market portfolio. The direction of rotation depends on which constraint dominates: clockwise for positive χ and counterclockwise for negative χ.*

E. Disagreement and Borrowing Constraints

Remark 6 *The net effect of borrowing constraints on CAPM is captured by χ in equation (12), the value of which depends on disagreement. In the following discussion we will show that (i) when disagreement is very low or very high, χ is close to zero and thus the standard CAPM holds; (ii) when disagreement is not extreme, leverage constraint is tighter than short-sales constraint, and thus $\chi > 0$. As highlighted in Remark 5, when χ is positive, the SML will rotate clockwise and thereby becomes flatter. If the net effect of borrowing constraint χ is large enough, the SML will be downward-sloping.*

The tightness of borrowing constraints, Ψ and Φ, depends on the level of disagreement. Optimists want to make leveraged investments in risky assets by borrowing cash from pessimists using loans collateralized by the assets they have. However, pessimists do not value the collateral as much as optimists do and thus are reluctant to lend, which leads to an endogenous constraint on optimists' ability to borrow, given their optimists' beliefs. Similarly, pessimists borrow assets from optimists to short sell using their cash holding as collateral. Since optimists expect an increase in asset prices, they would anticipate risks that pessimists' cash-collateral is deficient to repurchase the stocks they borrowed . Consequently, optimists tend to reduce assets available to lend relative to demand, and thus pessimists face short-sales constraints. In summary, investors would face limits on borrowing which resemble margin requirements.

Greater belief dispersion, i.e. the pessimists think that the asset is worth less and optimists think that it is worth more, increases borrowing demands so as to take positions in line with their expectations, which, in turn, tightens the borrowing constraints (ψ_i and ϕ_i increase). On the other hand, from the lenders' perspectives, the collateral becomes less valuable relative to the loan contract, which leads lenders to reducing lending, and hence the tightness of borrowing constraints increases (m^L and m^S increase). Overall, a higher belief dispersion increases demand for borrowing (demand-channel) and reduces supply of lending (supply-channel), and thus the borrowing constraints get tighter, for both cash and stocks.

The leverage constraint is generally tighter than the short-sales constraint in terms of the supply-channel effect. The intuition is as follows. Pessimistic lenders (of cash) discount the borrowers' collateral by a discount rate, which can be proxied by the ratio of the current asset price to pessimists' valuation, and the inverse ratio applies to optimistic lenders (of stocks to short-sellers). If pessimists and optimists expect the same absolute deviation from the current price, the discount rate used by pessimistic lenders of cash would be larger than that used by optimistic lenders of stocks, meaning that pessimistic lenders would discount the collateral more than optimistic lenders do, and thus the leverage constraint will be tighter than short-sales constraint (See Appendix C for a simple example illustrating this intuition).

When belief dispersion is very high, investors can hardly borrow cash or assets due to lenders' reluctance to lend either, which leads to the maximum supply-channel effect. As dispersion keeps increasing, the borrowing constraints will be dominated by the demand-channel effect, and consequently both leverage constraints and short-sales constraints will bind to a similarly high degree.

F. Disagreement and Trading Volume

Suppose for simplicity that each investor has initial endowment of risky assets $Z_{i,0} = \frac{\tau_i}{\tau} Z_m$ at time 0, which is a market portfolio weighted by the risk tolerance coefficient so that the sum of individual endowments is equal to the aggregate supply: $\sum_{i=1}^{I} Z_{i,0} = Z_m$. It can be shown that $Z_{i,0}$ is the optimal portfolio when there is no disagreement about the expected payoff by imposing $E_i(X) = E(X)$, $\Omega_i = \Omega$ and $\chi_i = 1$ in equation (10).

In the next period, investors have the identical assessment of the variance-covariance matrix but their beliefs about expected payoff start to diverge ($E_i(X) \neq E(X)$). Define investor i 's volume of trade in risky assets T_i as the actual demand (10) minus the endowment $Z_{i,0}$:

$$T_i = Z_i - Z_{i,0} = \tau_i \Omega^{-1} \left[E_i(X) - \chi_i E(X) + (\chi_i - 1) \tau^{-1} \Omega Z_m \right], \tag{14}$$

where $\chi_i = (R_f + \psi_i - \phi_i)(R_f + \Psi - \Phi)^{-1}$. Then the overall trading volume is given by the following corollary.

Corollary 7 *The equilibrium trading volume measure V is given by*

$$V = \frac{1}{2}\tau\Omega^{-1}\left[\triangle - (\Phi + \Psi) P\right],$$

where $\triangle = \sum_{i=1}^{I} \frac{\tau_i}{\tau} \mid E_i(X) - E(X) \mid$ *measures the dispersion in beliefs.*

PROOF: See Appendix D.

In the presence of high belief dispersion $\triangle$, trading volume tends to be high, as in Atmaz and Basak (2018); however, $(\Phi + \Psi)$, which increases with the level of disagreement, have negative effect on trading volume. Thus, disagreement generates two counteracting effects on trading volume: a positive direct effect characterized by the measure of belief dispersion; and a negative indirect effect through the shadow prices of borrowing constraints.

Although it is difficult to obtain an analytical derivative of trading volume with respect to disagreement, given the evidence (Atiase (1991), Bessembinder, Chan, and Seguin (1996), Goetzmann and Massa (2005)), we could plausibly conjecture that the positive direct effect is stronger in general. For example, consider a group of unconstrained optimists mildly who increase their expectation, which in turn should generate trading volume. Suppose they finance the increase in the long positions by borrowing more from pessimists who have excess cash holdings. Since the pessimists keep their expectation unchanged, they would regard the optimists' collateral as before. Therefore, the pessimists are very likely to provide the additional liquidity to optimists, who would remain unconstrained thereafter. Consequently, the increase in the optimists' expectations do not change the aggregate shadow costs of borrowing constraints, and thus the indirect channel of disagreement is shut down.

Remark 8 *The effect of disagreement on trading volume can be either positive or negative: trading volume increases in disagreement since investors with dispersed beliefs have high speculative demand; but disagreement also has a negative effect which limits the growth in trading volume, because the increase in disagreement tightens borrowing constraints and thus prohibits the amount of volume which would be traded if the constraints did not bind. In general, the positive effect is dominant, which is supported by several empirical studies. Overall, trading volume increases in* *disagreement.*

G. Disagreement and the Equity Premium

Rearranging (9) and normalizing the initial wealth of the market to one yields the equity premium

$$E\left(r_m\right) - r_f = \frac{1}{\tau}\sigma^2\left(r_m\right) + \left(\Psi - \Phi\right) \tag{15}$$

With the same expected market variance and risk-free rate as the economy with homogeneous beliefs, disagreement about expected payoffs leads to borrowing constraints binding for some investors, and the leverage constraint is generally tighter than the short-sales constraint (as discussed in Remark 6). Thus, $\Psi - \Phi$ is positive and increases the premium. As belief dispersion keeps increasing, the two constraints tend to bind to a similar extent and thus the net effect approaches zero.

Moreover, high disagreement might be associated with higher expected market variance ($\sigma^2\left(r_m\right)$), which is the implication of Banerjee and Kremer (2010) following the assumption that investors' uncertainty about the interpretation of future payoff increases with the level of disagreement. The positive relation between belief dispersion and expected market variance is also confirmed empirically by Barinov (2013). A plausible explanation for this positive relation might be that, when the belief is highly dispersed in the stock market, investors would observe some other investors having very different beliefs, which makes them less confident about their own interpretation, that is, the expected market variance increases. Equation (15) shows that this increase of investor confidence risk would be priced in the equity premium, as in Bansal and Shaliastovich (2010).[i]

Remark 9 *In sum, heterogeneous beliefs command a positive equity premium, either through increasing the shadow prices of borrowing constraints, or through decreasing investors' confidence (increasing the expected market variance).*

[i]This is not allowed in the model, which has dogmatic differences in beliefs. A more general model can allow for these effects with a considerable increase in complexity.

H. Model Predictions

Our model generates the following testable predictions:

(i) A high level of disagreement is associated with high trading volume and equity premium (Remark 8 and Remark 9);

(ii) On very low- (or high-) disagreement days, the net effect of two borrowing constraints is close to zero, and thus the standard CAPM will apply (Remark 6: (i)).

(iii) When disagreement is moderate, leverage constraint tends to be tighter than short-sales constraint. As a consequence, *distance-to-CAPM* is positive for small beta ($\beta < 1$) and negative for large beta ($\beta > 1$). The empirical SML is above that predicted by the standard CAPM for low-beta assets and below it for high-beta assets (Remark 6: (ii)).

We next provide evidence for our model predictions.

III. Empirical Analysis

A. Data

Our measure of disagreement is the daily trading volume of all listed stocks on New York Stock Exchange, American Stock Exchange, the Nasdaq Stock Market and the Arca Stock Market. The volume is defined as in Campbell, Grossman, and Wang (1993), which is the log ratio of number of shares traded to the number of outstanding, detrended by a one-year moving average filter. We obtain trading-volume data from the Center for Research in Security Prices (CRSP). Figure 1 plots the time series of our disagreement measure.

We study the relation between CAPM betas and return using ten beta-sorted portfolios provided by Savor and Wilson (2014). They first estimate stock market betas for all stocks on CRSP using rolling windows of 12 months of daily returns from 1964 to 2011, and then sort stocks into one of ten beta-decile value-weighted portfolios. Following Savor and Wilson (2014), our stock market proxy is the CRSP NYSE, Amex, and Nasdaq value-weighted index of all listed shares. Table 1 presents descriptive statistics for those variables.

B. Results

We evaluate our model implications as follows. First, we sort *Turnover* into three groups: very low-disagreement days (below the 10th percentile of *Turnover*), moderate-disagreement days (range from 10th percentile to the 90th percentile) and very high-disagreement days (above the 90th percentile). We then separately plot the SML for periods of low, moderate, and high disagreement. The results are summarized in Figure 2- 4.

Prediction 1 *A high level of disagreement is associated with high equity premium.*

The average excess market return $\bar{r}^e_M$ is -9.8, 1.3, and 19.9 basis points (bps), respectively, for low, moderate, and high disagreement. This shows that equity premium increases with the level of disagreement, which is consistent with Prediction 1.

Prediction 2 *On very low or very high disagreement days, the standard CAPM holds.*

During the periods of very low or high disagreement (Figure 2 and Figure 4), the empirical SML (solid points and a solid line) runs very near its CAPM prediction (a dotted line).[ii] Note that the SML on very low-disagreement days is downward-sloping; however, it does not indicate the rejection of the CAPM because the average excess return one low-disagreement days is negative. The R^2s of SMLs are 98% for low-disagreement days and 99% for high-disagreement days, respectively, indicating that most of the variation in average excess returns is accounted for by their stock market betas. This evidence is consistent with Prediction 2.

Prediction 3 *When disagreement is not extreme, the empirical SML is above that predicted by the standard CAPM for low-beta assets ($\beta < 1$) and below it for high-beta assets ($\beta > 1$).*

[ii] On low-disagreement days, the intercept is 0.1 bps and is not significant different from zero (t-statistics=0.2). The slope of the SML is -10.5 bps (t-statistics=-18.9), and it is not significantly different from the average market excess return of -9.8 bps (the t-statistics for the difference is 1.3). On high-disagreement days, the intercept is -2.3 bps and is just marginally significantly different from zero (t -statistics=-2.5). The slope of the SML is 22.6 bps (t -statistics=23.2), and it is just marginally significantly different from the average market excess return of 19.9 bps (the t-statistics for the difference is 2.8).

When disagreement is moderate (Figure 3), the empirical SML (solid points and a solid line) differs significantly from that predicted by the standard CAPM (a dotted line), and its R^2 is 54%.[iii] Furthermore, average excess returns are plotted above the CAPM prediction for $\beta < 1$, and below it for $\beta > 1$, supporting Prediction 3.

IV. Conclusion

This paper shows that the interaction of heterogeneous expectations, leverage constraints and short-sales constraints determines the Security Market Line, both theoretically and empirically. Disagreement drives borrowing demand for both cash and risky assets: optimists who think assets are underpriced may want to borrow money from pessimists to take more positions; in contrast, pessimists may want to borrow more assets from optimists to sell short. However, disagreement reduces the supply of loans because lenders do not value the collateral which consists of borrowers' wealth as much as their borrowing counterparts do, and therefore are reluctant to meet demand fully. This leads to endogenous leverage and short-sales constraints, both of which arise in equilibrium. In equilibrium, a modified version of CAPM holds which incorporates the shadow prices of these borrowing constraints.

[iii] On moderate-disagreement days, the intercept is 2.4 bps and is significantly different from zero (t-statistics=6.6). The slope of the SML is -1.2 bps (t-statistics=-3.1), and it is significantly different from the average market excess return of 1.3 bps (the t-statistics for the difference is 6.5).

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Tables and Figures

Table 1. Summary Statistics for Time-Series Variables

This table presents descriptive statistics for key variables used in the paper. *Turnover* is the measure of disagreement. *MKTRF* is the excess return of the CRSP value-weighted index. *R(1)-R(10)* are the returns of ten beta-sorted portfolios. Returns are expressed in percent. The sample period is 1/2/1964 to 12/30/2011 with 12085 daily observations.

	Mean	Std.Dev.	P10	P25	Median	P75	P90
Turnover	0.031	0.238	-0.248	-0.107	0.029	0.164	0.428
MKTRF	0.021	1.008	-1.050	-0.430	0.050	0.490	1.473
R(1)	0.031	0.798	-0.692	-0.251	0.050	0.337	1.107
R(2)	0.044	0.592	-0.533	-0.201	0.060	0.309	0.866
R(3)	0.043	0.605	-0.562	-0.206	0.066	0.315	0.872
R(4)	0.039	0.682	-0.651	-0.243	0.066	0.349	0.993
R(5)	0.047	0.776	-0.736	-0.287	0.076	0.404	1.090
R(6)	0.042	0.894	-0.868	-0.339	0.069	0.454	1.264
R(7)	0.043	1.029	-1.008	-0.412	0.066	0.515	1.496
R(8)	0.039	1.165	-1.162	-0.480	0.064	0.573	1.704
R(9)	0.038	1.417	-1.414	-0.605	0.056	0.690	2.055
R(10)	0.036	1.900	-1.957	-0.833	0.074	0.901	2.734

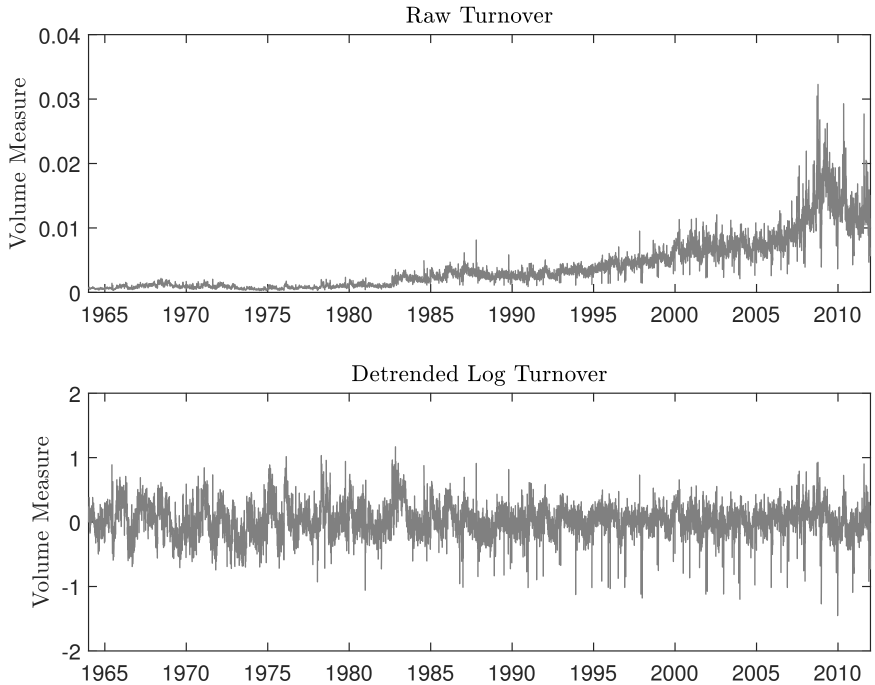

Figure 1. Stock Market Turnover

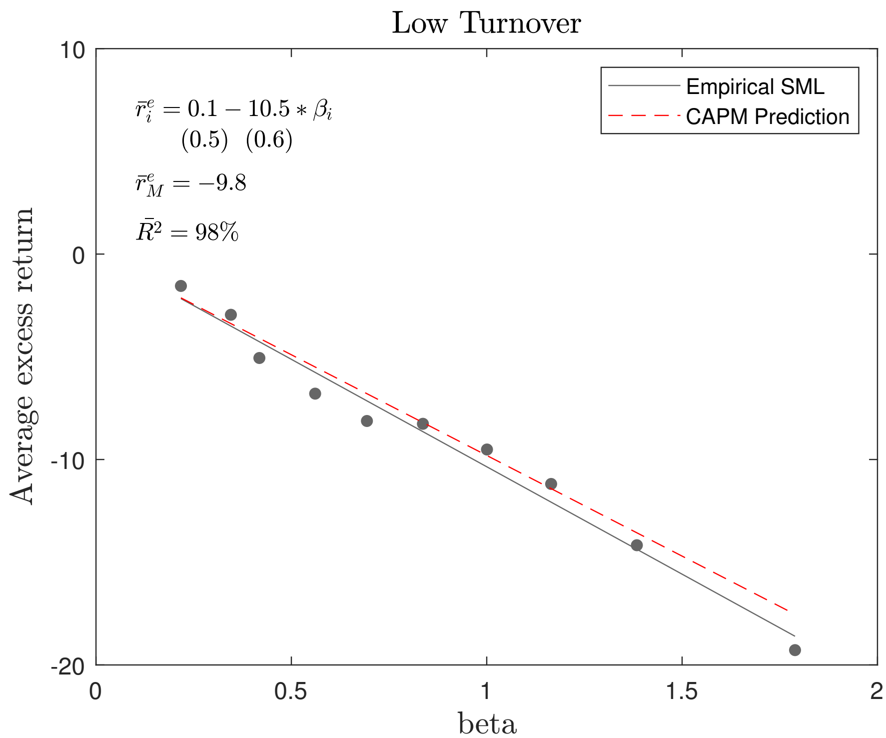

Figure 2. SML on Low-turnover Days. This figure plots average daily excess returns in bps against market betas for ten beta-sorted portfolios of all NYSE, Amex, and Nasdaq stocks (the solid points), and the implied ordinary least squares estimates of the SML (the solid line). The sample covers the 1964–2011 period. In addition, the SML predicted by the standard CAPM is displayed (the dotted line).

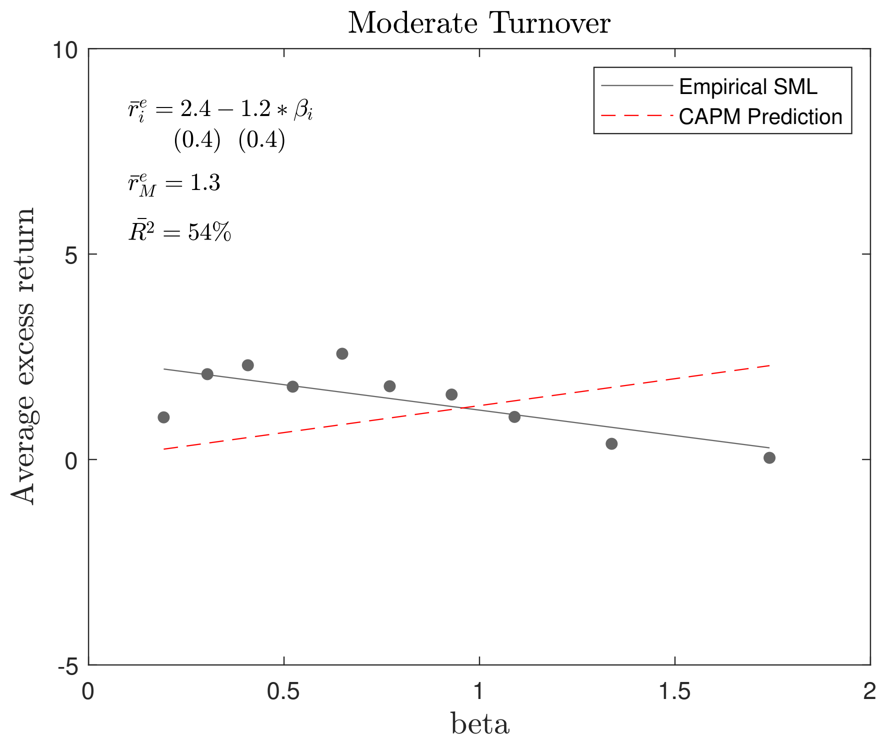

Figure 3. SML on Moderate-turnover Days. This figure plots average daily excess returns in bps against market betas for ten beta-sorted portfolios of all NYSE, Amex, and Nasdaq stocks (the solid points), and the implied ordinary least squares estimates of the SML (the solid line). The sample covers the 1964–2011 period. In addition, the SML predicted by the standard CAPM is displayed (the dotted line).

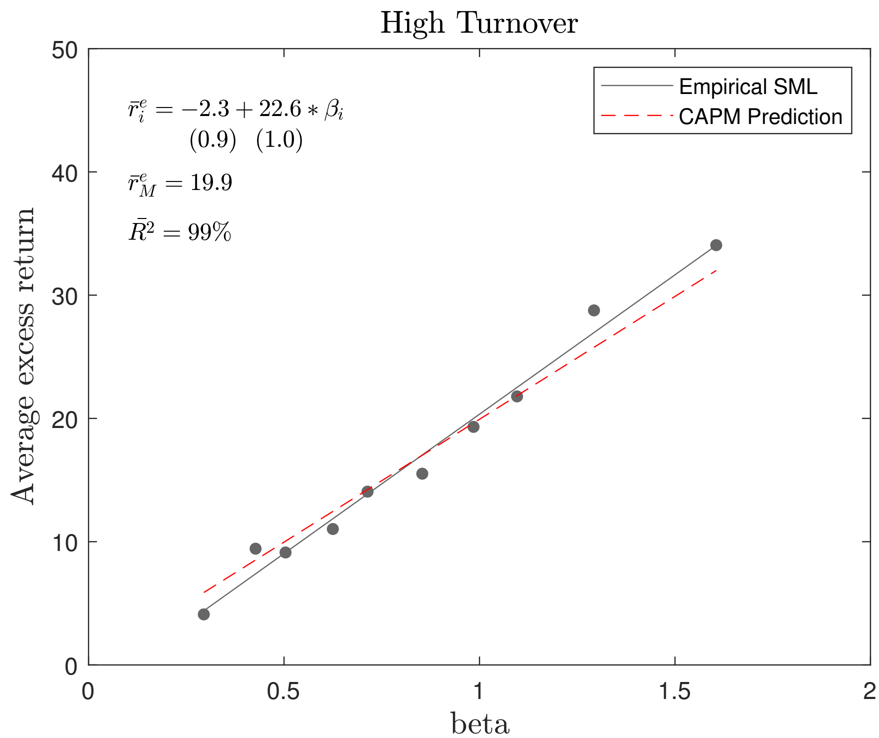

Figure 4. SML on High-turnover Days. This figure plots average daily excess returns in bps against market betas for ten beta-sorted portfolios of all NYSE, Amex, and Nasdaq stocks (the solid points), and the implied ordinary least squares estimates of the SML (the solid line). The sample covers the 1964–2011 period. In addition, the SML predicted by the standard CAPM is displayed (the dotted line).

Appendices

Appendix A: Proof of Lemma 2

Substituting investors' asset demands (4) into the market clearing condition (5) yields

$$Z_m = \sum_{i=1}^{I} Z_i = \sum_{i=1}^{I} \tau_i \Omega_i^{-1} \left[E_i(X) - \left(R_f + m^L \psi_i - m^S \phi_i \right) P \right] \quad (16)$$

Using Definition 1, we rewrite (16) as

$$Z_m = \tau \Omega^{-1} \left[E(X) - \left(R_f I_K + \Psi - \Phi \right) P \right] \quad (17)$$

Rearranging (17) leads to

$$P = \left(R_f I_K + \Psi - \Phi \right)^{-1} \left[E(X) - \tau^{-1} \Omega Z_m \right].$$

Substituting P into the demand function (4) gives the optimal demand of investor i:

$$Z_i = \tau_i \Omega_i^{-1} \left[E_i(X) - \chi_i \left(E(X) - \tau^{-1} \Omega Z_m \right) \right].$$

where $\chi_i = \left(R_f + \psi_i - \phi_i \right) \left(R_f I_K + \Psi - \Phi \right)^{-1}$.

■

Appendix B: Proof of Proposition 3

Under consensus belief the the variance of market portfolio is expressed as $\sigma^2(W_m) = Z_m^T \Omega Z_m$. Substituting (17) into $\sigma^2(W_m)$ gives

$$\sigma^2(W_m) = \tau \left[E(W_m) - R_f W_{m,0} - \Psi_m + \Phi_m \right] \quad (18)$$

where $W_{m,0} = P^T Z_m$, $\Psi_m = P^T \Psi Z_m$ and $\Phi_m = P^T \Phi Z_m$. Rearranging (18) gives

$$\frac{1}{\tau} = \frac{E(W_m) - R_f W_{m,0} - \Psi_m + \Phi_m}{\sigma^2(W_m)}. \quad (19)$$

Substituting (19) into (9) yields

$$E(X) - R_f P = \frac{1}{\sigma^2(W_m)} \Omega Z_m \left[E(W_m) - R_f W_{m,0} - \Psi_m + \Phi_m \right] + (\Psi - \Phi) P,$$

or equivalently

$$E(x_k) - R_f p_k = \frac{\sigma(W_m, x_k)}{\sigma^2(W_m)} \left[E(W_m) - R_f W_{m,0} - \Psi_m + \Phi_m \right] + ((\Psi - \Phi) P)_k . \tag{20}$$

Dividing (20) throughout by p_k gives

$$E(r_k) - r_f = \beta_k \left[E(r_m) - r_f \right] + \frac{[(\Psi - \Phi) P]_k}{p_k} - \beta_k \frac{P^T (\Psi - \Phi) Z_m}{P^T Z_m}. \tag{21}$$

■

Appendix C: An example for Remark 6

To exemplify the intuition behind Remark 6, consider a simple example with one asset, two periods (0,1), and two investors who disagree on the future price of the asset. Suppose the price of the asset per unit at date 0 is P. The optimist (Anna) believes the price will increase by Δ_A at date 1 and the pessimist (Bob) expects a decrease Δ_B. Suppose both of them have identical wealth W, which consists only of cash for Anna and the risky asset for Bob. Given their expectations, Anna will borrow cash from Bob to buy more shares of the asset; Bob will borrow the asset from Anna to short sell. Assume their optimal amounts of borrowing in terms of cash are identically equal to W; however, their demands for loans may not be satisfied due to the borrowing constraints.

Suppose Anna borrows L dollars of cash using W/P units of asset as collateral at date 0, and promises to repay L dollars at date 1, assuming she can borrow at zero cost. Since Bob believes the asset price at date 1 is $P - \Delta_B$ and hence expects the value of Anna's collateral at date 1 is $(P - \Delta_B) W/P$. Bob is willing to lend only if the expected value of collateral is greater or equal to repayment. Thus, the maximum loan amount is $L = W(P - \Delta_B)/P$.

Similarly, Bob borrows S/P units of the asset using W dollars of cash as collateral at date 0 and promises to repay S/P units of the asset at date 1. Since Anna believes the asset price at date 1 is $P + \Delta_A$, and therefore expects the value of repayment at date 1 is $(P + \Delta_A) S/P$. Anna is willing to lend only if the expected value of collateral is greater or equal to repayment. Thus the maximum loan amount is $S = WP/(P + \Delta_A)$.

Define the tightness of borrowing constraints as the fraction of demands for loans which are not satisfied. Let C^L and C^S denote the tightness of leverage constraint and short-sales constraint, respectively

$$C^L = 1 - \frac{L}{W} = \frac{\Delta_B}{P} = \delta_B,$$

and

$$C^S = 1 - \frac{S}{W} = \frac{\Delta_A/P}{1 + \Delta_A/P} = \frac{\delta_A}{1 + \delta_A},$$

where $\delta_{A(B)} = \Delta_{A(B)}/P$ is the rate of return on the asset expected by Anna (Bob). δ_B ranges from 0 to 1 because the largest price decline expected by Bob is P, while δ_A ranges from 0 to positive infinity because there is no upper bound for the optimistic expectation. As illustrated in Figure 5, C^L and C^S are both monotonic functions of δ. When there is no disagreement, lenders evaluate borrowers as wealthy as themselves and thus are willing to lend. Consequently, borrowers can borrow the full amount with the borrowing constraint not binding. An increase in disagreement leads to an increase in the discount of the collateral value by lenders, and hence lenders are more reluctant to lend and tighten the borrowing constraints.

With the same level of δ, the leverage constraint is always tighter than the short-sales constraint. When pessimists expect the asset to be worthless ($\delta_B = 1$) and thus the collateral value to be zero, they will not lend cash at all and $C^L = 1$. As optimism increases to a very high level, optimists will not lend assets out because they think pessimists cannot afford to buy back the asset which was sold short, which leads to C^S approaching one. In other words, when belief

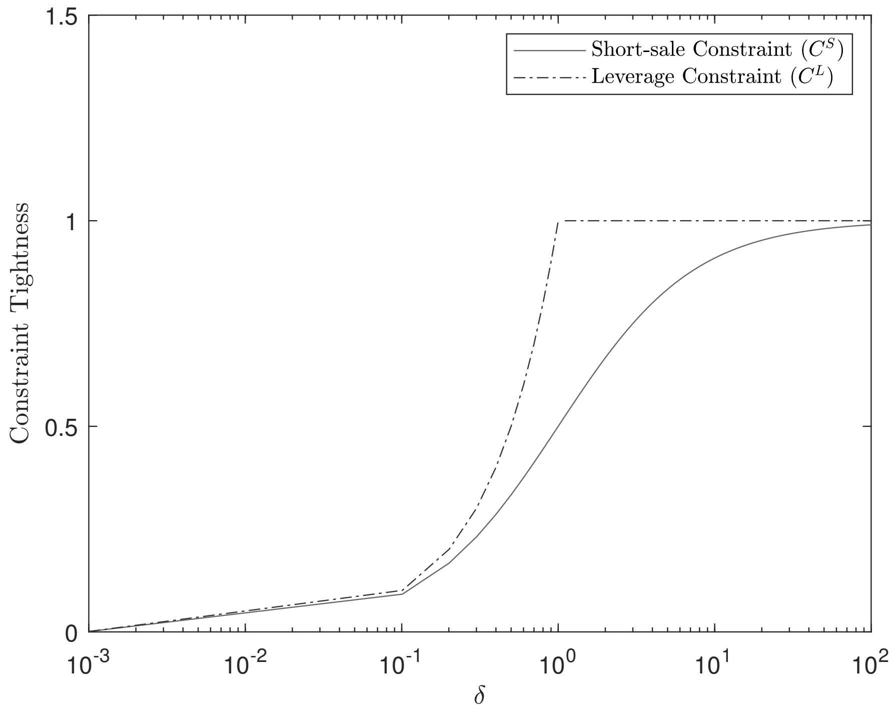

Figure 5. Borrowing Constraints

dispersion is very high, both the leverage constraint and the short-sales constraint will bind to a similarly large degree.

Appendix D: Proof of Corollary 7

We rewrite investor i's expectation as $E_i(X) = E(X) + \triangle_i$, where $\triangle_i$ is the investor i's bias . Substitute this expression of $E_i(X)$ into (14) yields

$$T_i = \tau_i \Omega^{-1} \left[(1 - \chi_i) \left(E(X) - \tau^{-1} \Omega Z_m \right) + \triangle_i \right]. \tag{22}$$

Plugging (9) into (22) yields

$$T_i = \tau_i \Omega^{-1} \left[(\Psi - \Phi - \psi_i + \phi_i) P + \triangle_i \right]. \tag{23}$$

Now consider an investor who has the correct belief and thus $\triangle_i = 0$. For simplicity we assume the investor is not constrained, $\psi_i = \phi_i = 0$, and thus $T_i = \tau_i (\Psi - \Phi) \Omega^{-1} P$. Therefore, the investor holds a portfolio that deviates from the market portfolio due to the aggregate effect of borrowing constraints for other investors who hold biased beliefs. Consequently, the market portfolio is no more the tangency portfolio. We further assume that optimistic (pessimistic) investors have positive (negative) biases for all risky assets, and the number of shares traded in each asset is nonnegative (nonpositive), i.e. $\triangle_{i,k} > 0$ ($\triangle_{i,k} < 0$) and $T_{i,k} \geq 0$ ($T_{i,k} \leq 0$) for $k = 1, \cdots, K$.

First consider unconstrained investors for whom the shadow prices of borrowing constraints are zero: $\psi_i = \phi_i = 0$. Thus, his trading volume becomes $T_i = \tau_i \Omega^{-1} \left[(\Psi - \Phi) P + \triangle_i \right]$. Suppose each individual has negligible influence on the overall tightness of constraints. All else equal, an increase in the absolute value of investor's bias leads to a higher trading volume. Next, consider the leverage (short-sales) constrained investors. Note that constrained investors' dollar volume is determined by the margin requirement and their initial wealth. Let D_i^L (D_i^S) be the dollar trading volume for a leverage (short-sales) constrained investor:

$$D_i^L = W_{i,0}/m^L - (W_{i,0} - z_f) = P^T T_i$$

$$D_i^S = W_{i,0}/m^S + (W_{i,0} - z_f) = -P^T T_i,$$

which together with (23) yields ψ_i and ϕ_i

$$\psi_i = (\Psi - \Phi) + \frac{P^T \Omega^{-1} \triangle_i - D_i^L/\tau_i}{P^T \Omega^{-1} P}$$

$$\phi_i = (\Phi - \Psi) - \frac{P^T \Omega^{-1} \triangle_i + D_i^S/\tau_i}{P^T \Omega^{-1} P}.$$

Then the overall trading volume is measured as the sum of the absolute trading volume of all investors:

$$V = \frac{1}{2}\sum_{i=1}^{I} \mid T_i \mid,$$

where the adjustment 1/2 prevents double summation of the shares across investors. Investors are naturally divided into four groups: (i) unconstrained optimists $i \in U+$; (ii) unconstrained pessimists $i \in U-$; (iii) constrained optimists $i \in C+$; and (iv) constrained pessimists $i \in C-$. Suppose four groups are of equal importance in determining the consensus belief, i.e. $\sum\limits_{i\in U+}\tau_i = \sum\limits_{i\in U-}\tau_i = \sum\limits_{i\in C+}\tau_i = \sum\limits_{i\in C-}\tau_i$. Then the trading volume can be shown to be

$$\begin{aligned}
V &= \tfrac{1}{2}\left(\sum_{i\in U+} \mid T_i \mid + \sum_{i\in U-} \mid T_i \mid + \sum_{i\in C+} \mid T_i \mid + \sum_{i\in C-} \mid T_i \mid\right) \\
&= \tfrac{1}{2}\Big(\sum_{i\in U+}\tau_i \Omega^{-1}\left[(\Psi-\Phi)P + \triangle_i\right] + \sum_{i\in U-}\tau_i\Omega^{-1}\left[-(\Psi-\Phi)P - \triangle_i\right]\cdots \\
&\quad + \sum_{i\in C+}\tau_i\Omega^{-1}\left[(\Psi-\Phi)P + \triangle_i - \psi_i P\right] + \sum_{i\in C-}\tau_i\Omega^{-1}\left[-(\Psi-\Phi)P - \triangle_i - \phi_i P\right]\Big) \\
&= \tfrac{1}{2}\Omega^{-1}\left(\textstyle\sum_{i=1}^{I}\tau_i \mid \triangle_i \mid - \displaystyle\sum_{i\in C+}\tau_i\psi_i P - \sum_{i\in C-}\tau_i\phi_i P\right) \\
&= \tfrac{1}{2}\tau\Omega^{-1}\left[\Delta - (\Psi+\Phi)P\right]
\end{aligned}$$

where $\triangle = \sum_{i=1}^{I}\frac{\tau_i}{\tau} \mid \triangle_i \mid$ measures the dispersion in beliefs. ■